



1.6.3 Complex numbers and their components





Introduction

- In this topic, we will
 - Introduce complex numbers
 - Define the real and imaginary components
 - Define real and imaginary complex numbers
 - Describe the rectangular representation and preferences
 - Define equality of two complex numbers





j and the square root

- In the last topic, we introduced $j \stackrel{\text{def}}{=} \sqrt{-1}$ so $j^2 = -1$
 - In taking the square root of a negative number:

$$\sqrt{-100} = \sqrt{100 \cdot (-1)} = \sqrt{100} \sqrt{-1} = 10j$$

- Recall, if $\alpha > 0$ then $\sqrt{\alpha} > 0$
 - Thus, if $\alpha > 0$ then

$$\sqrt{\alpha\beta} = \sqrt{\alpha} \sqrt{\beta}$$

- Note that we will see, in general, $\sqrt{\alpha\beta} \neq \sqrt{\alpha} \sqrt{\beta}$
 - You can only do this if $\alpha > 0$ or $\beta > 0$
 - For example,

$$10 = \sqrt{100} = \sqrt{(-10)(-10)} \neq \sqrt{-10} \sqrt{-10} = \sqrt{10} j \sqrt{10} j = \sqrt{10}^2 j^2 = -10$$





j and the square root

- Thus, what are the roots of $x^2 - 5.4x + 16.9$

$$\begin{aligned}\frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{5.4 \pm \sqrt{29.16 - 67.6}}{2} \\ &= \frac{5.4 \pm \sqrt{-36.44}}{2} \\ &= \frac{5.4}{2} \pm \frac{\sqrt{36.44 \cdot (-1)}}{2} \\ &= 2.7 \pm \frac{\sqrt{36.44} \sqrt{-1}}{2} \\ &= 2.7 \pm \frac{6.2j}{2} = 2.7 \pm 3.1j\end{aligned}$$





The set of complex numbers $\mathbf{C}$

- We will define a complex number as any number that can be written in the form

$$\alpha + \beta j$$

where both α and β are real numbers

- Recall that we represent the set of all real numbers by $\mathbf{R}$

- We will represent the set of all complex numbers by $\mathbf{C}$

$$\mathbf{C} \stackrel{\text{def}}{=} \{ \alpha + \beta j : \alpha \in \mathbf{R} \wedge \beta \in \mathbf{R} \}$$

such that α is a member of the reals and β is a member of the reals

α is a member of the reals





The real and imaginary components

- Suppose

$$z = \alpha + \beta j \in \mathbb{C}$$

where α and β are real

- We will define:

- The *real component* of z as

$$\Re(z) = \Re(\alpha + \beta j) \stackrel{\text{def}}{=} \alpha$$

- The *imaginary component* of z as

$$\Im(z) = \Im(\alpha + \beta j) \stackrel{\text{def}}{=} \beta$$

- Note that the imaginary component is a real number!





The real and imaginary components

- For example,

- If $z = 3.2 - 5.7j$, then

$$\Re(z) = 3.2$$

$$\Im(z) = -5.7$$

- If $w = -4.8 + 1.9j$, then

$$\Re(w) = -4.8$$

$$\Im(w) = 1.9$$





Real, imaginary and purely imaginary

- A complex number such that $\Im(z) = 0$, that is

$$z = \alpha + 0j,$$

is said to be *real*

- A complex number such that $\Re(z) = 0$, that is

$$z = 0 + \beta j,$$

is said to be *imaginary*

Note that $0 + 0j$ is both real and imaginary





Rectangular representation

- When a complex number is written in the form

$$\alpha + \beta j$$

where α and β are real, is the *rectangular representation*

- Examples:

$$-5.4 - 3.1j$$

$$45.5 + 71.32j$$

$$\sin(1.8) - \sqrt{2.9}j$$

$$6.323 - 5.910 \times 10^{-8}j$$

- $\sqrt{-1.44}$ is complex, but its rectangular representation is $1.2j$





Preferred representations

- We have preferred forms for writing some complex numbers:

Description	Preferred representation	Other rectangular representations
Zero	0	$0 + 0j$ $0j$ $0 + j0$ $j0$
Real	α	$\alpha + 0j$ $\alpha + j0$
Imaginary	βj	$0 + \beta j$ $0 + j\beta$ $j\beta$
Negative imaginary component ($\beta > 0$)	$\alpha - \beta j$	$\alpha + (-\beta)j$ $\alpha + j(-\beta)$ $\alpha - j\beta$

We prefer $3.2 + 4.7j$ because
in MATLAB, you would write
`>> 3.2 + 4.7j`





Equality

- Two complex numbers z and w are equal if and only if

$$\Re(z) = \Re(w) \text{ and } \Im(z) = \Im(w)$$

- In other words, if $z = \alpha + \beta j$ and $w = \gamma + \delta j$,
then $z = w$ if and only if

$$\alpha = \gamma \text{ and } \beta = \delta$$

- Examples:

$$3.2 + 5.7j \neq -2.8 + 5.7j$$

$$3.2 + 5.7j \neq 3.2 + 9.1j$$

$$3.2 + 5.7j \neq 3.2$$

$$3.2 + 5.7j \neq 5.7j$$





Summary

- In this topic, we've introduced complex numbers
 - Defined the set of complex number $\mathbf{C}$
 - Defined the real and imaginary components
 - Real numbers have zero imaginary component
 - Imaginary numbers have zero real component
 - Described the rectangular representation
 - There are preferred representations: 3.2 , $4.7 - 2.9j$, $5.1j$
 - Defined equality of two complex numbers





References

- [1] https://en.wikipedia.org/wiki/Complex_numbers
- [2] https://en.wikipedia.org/wiki/Imaginary_number





Acknowledgments

None so far.





Colophon

These slides were prepared using the Cambria typeface. Mathematical equations use Times New Roman, and source code is presented using Consolas.

The photographs of flowers and a monarch butter appearing on the title slide and accenting the top of each other slide were taken at the Royal Botanical Gardens in October of 2017 by Douglas Wilhelm Harder. Please see

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